

Phase transitions in the time synchronization model

V. Malyshev* A. Manita

Date: manuscript — August 2004,
 ArXiv — January 2012

Abstract

We continue the study of the time synchronization model from arXiv:1201.2141 . There are two types $i = 1, 2$ of particles on the line \mathbb{R} , with N_i particles of type i . Each particle of type i moves with constant velocity v_i . Moreover, any particle of type $i = 1, 2$ jumps to any particle of type $j = 1, 2$ with rates $N_j^{-1}\alpha_{ij}$. We find phase transitions in the clusterization (synchronization) behaviour of this system of particles on different time scales $t = t(N)$ relative to $N = N_1 + N_2$.

Keywords: Markov process, stochastic particles system, synchronization model

2000 MSC: 60K35, 60J27, 60F99

1 The Model and Main result

The simplest formulation of the model, we consider here, is in terms of the particle system. On the real line there are N_1 particles of type 1 and N_2 particles of type 2, $N = N_1 + N_2$. Each particle of type $i = 1, 2$ performs two independent movements. First of all, it moves with constant speed v_i in the positive direction. We assume further that v_i are constant and different, thus we can assume without loss of generality that $0 \leq v_1 < v_2$. The degenerate case $v_1 = v_2$ is different and will be considered separately.

Secondly, at any time interval $[t, t + dt]$ each particle of type i independently of the others with probability $\alpha_{ij}dt$ decides to make a jump to some particle of type j and chooses the coordinate of the j -type particle, where to jump, among the particles of type j , with probability $\frac{1}{N_j}$. Here α_{ij} are given nonnegative parameters for $i, j = 1, 2$. Further on, unless otherwise stated, we assume that $\alpha_{11} = \alpha_{22} = 0$, $\alpha_{12}, \alpha_{21} > 0$.

*Postal address: Faculty of Mathematics and Mechanics, Moscow State University, Leninskie Gory 1, GSP-1, 119991, Moscow, Russia. E-mail: malyshev2@yahoo.com, manita@mech.math.msu.su .

First published in Theory Probab. Appl. 50, pp. 134-141 (2006)

After such instantaneous jump the particle of type i continues the movement with the same velocity v_i . This defines continuous time Markov chain

$$\xi_{N_1, N_2}(t) = \left(x_1^{(1)}(t), \dots, x_{N_1}^{(1)}(t); x_1^{(2)}(t), \dots, x_{N_2}^{(2)}(t) \right), \quad (1)$$

where $x_k^{(i)}(t)$ is the coordinate of k -th particle of type i at time t . We assume that the initial coordinates $x_k^{(i)}(0)$ of the particles at time 0 are given. We are interested in the long time evolution of this system on various scales with $N \rightarrow \infty$, $t = t(N) \rightarrow \infty$.

In different terms, this can be interpreted as the time synchronization problem. In general, time synchronization problem can be presented as follows. There are N systems (processors, units, persons etc.) There is an absolute (physical) time t , but each processor j fulfills a homogeneous job in its own proper time $t_j = v_j t$, $v_j > 0$. Proper time is measured by the amount v_j of the job, accomplished by the processor for the unit of the physical time, if it is disjoint from other processors. However, there is a communication between each pair of processors, which should lead to drastic change of their proper times. In our case the coordinates $x_k^{(i)}(t)$ can be interpreted as the modified proper times of the particles-processors, the nonmodified proper time being $x_k^{(i)}(0) + v_i t$.

There can be many variants of exact formulation of such problem, see [1, 4, 6]. We will call the model considered here the basic model, because there are no restrictions on the jump process. Many other problems include such restrictions, for example, only jumps to the left are allowed. Due to absence of restrictions, this problem, as we will see below, is a “linear problem” in the sense that after scalings it leads to linear equations. In despite of this it has nontrivial behaviour, one sees different picture on different time scales.

There are, however, other interesting interpretations of this model, related to psychology, biology and physics. For example, in social psychology perception of time and life tempo strongly depends on the social contacts and intercourse. We will not enter the details here.

We show that the process consists of three consecutive stages: initial desynchronization up to the critical scale, critical slow down of desynchronization and final stabilization.

Introduce the empirical means (mass centres) and the empirical variances

$$\overline{x^{(i)}}(t) = \frac{1}{N_i} \sum_{k=1}^{N_i} x_k^{(i)}(t), \quad S_i^2(t) = \frac{1}{N_i} \sum_{k=1}^{N_i} \left(x_k^{(i)}(t) - \overline{x^{(i)}}(t) \right)^2$$

for types 1 and 2 and their means

$$\boldsymbol{\mu}_i(t) = \overline{\mathbf{E}x^{(i)}}(t), \quad \mathbf{l}_{12}(t) = \boldsymbol{\mu}_1(t) - \boldsymbol{\mu}_2(t), \quad R_i(t) = \mathbf{E}S_i^2(t)$$

Our first result concerns the asymptotic behavior of the empirical means.

Theorem 1 *For any sequences of triplets (N_1, N_2, t) such that $\min(N_1, N_2) \rightarrow \infty$ and $t = t(N_1, N_2) \rightarrow \infty$ the following statements hold*

$$\mathbf{l}_{12}(t) \rightarrow \frac{v_1 - v_2}{\alpha_{12} + \alpha_{21}}, \quad \frac{\boldsymbol{\mu}_i(t)}{t} \rightarrow \frac{\alpha_{12}v_2 + \alpha_{21}v_1}{\alpha_{12} + \alpha_{21}}$$

Assume now that $N_i = [c_i N]$, where $c_i > 0$, $c_1 + c_2 = 1$. Results of the next theorem cover all domain of asymptotical behavior of $t(N)$ as $N \rightarrow \infty$.

Theorem 2 *There are the following three regions of asymptotic behaviour, uniform in $t(N)$ for sufficiently large N :*

- if $\frac{t(N)}{N} \rightarrow 0$, then $R_i(t(N)) \sim h \varkappa_2 t(N)$,
- if $t = t(N) = sN$ for some $s > 0$, then $R_i(t(N)) \sim h(1 - e^{-\varkappa_2 s})N$,
- if $\frac{t(N)}{N} \rightarrow \infty$, then $R_i(t(N)) \sim hN$,

where the constant $\varkappa_2 > 0$ is defined by the formula (11), see below, and

$$h = \frac{2\alpha_{12}\alpha_{21}(v_1 - v_2)^2}{\varkappa_2(\alpha_{12} + \alpha_{21})^3}.$$

2 Proof

Embedded Markov Chain To prove theorems 1 and 2 we will use an embedded Markov chain. Consider the continuous time Markov chain $\xi_{N_1, N_2}(t) = \xi_{N_1, N_2}(t, \omega)$, defined by (1), and let

$$\tau_1(\omega) < \tau_2(\omega) < \dots < \tau_n = \tau_n(\omega) < \dots$$

be the random moments of particle jumps. Then $\{\tau_{n+1} - \tau_n\}_{n=1}^{\infty}$ are i. i. d. random variables exponentially distributed with mean $\gamma_{N_1, N_2} = (N_1\alpha_{12} + N_2\alpha_{21})^{-1}$.

We introduce a *discrete time* Markov chain $\zeta_{N_1, N_2}(n)$, $n = 1, 2, \dots$,

$$\zeta_{N_1, N_2}(n, \omega) = \xi_{N_1, N_2}(\tau_n(\omega), \omega)$$

with the same state space $\mathbb{R}^{N_1+N_2}$. The idea is that the asymptotic behavior of the continuous time particle system $\xi_{N_1, N_2}(t)$ can be reduced to the asymptotic properties of the discrete time chain $\zeta_{N_1, N_2}(n)$. Indeed, by the Law of Large Numbers

$$\tau_n \sim \frac{n}{N_1\alpha_{12} + N_2\alpha_{21}} \quad (n \rightarrow \infty)$$

In other words, if n is large, the value $n\gamma_{N_1, N_2} = n(N_1\alpha_{12} + N_2\alpha_{21})^{-1}$ is asymptotically equal to the “physical” time t associated with the continuous time particle system $\xi_{N_1, N_2}(t)$. Similarly to the empirical mean and empirical variance we introduce, for the embedded chain,

$$X_i(n) = \frac{1}{N_i} \sum_{k=1}^{N_i} x_k^{(i)}(\tau_n), \quad D_i(n) = \frac{1}{N_i} \sum_{k=1}^{N_i} \left(x_k^{(i)}(\tau_n) - X_i(n) \right)^2$$

Evidently, $X_i(n) = \overline{x^{(i)}}(\tau_n)$ and $D_i(n) = S_i^2(\tau_n)$. In the sequel we shall deal with their expected values

$$\mu_i(n) = \mathbb{E}X_i(n), \quad d_i(n) = \mathbb{E}D_i(n),$$

and shall need also the notation

$$l_{12}(n) = \mu_1(n) - \mu_2(n), \quad r(n) = \mathbb{E}(X_1(n) - X_2(n))^2$$

Closed equation for empirical means Here we prove Theorem 1.

The following lemma can be checked by a straightforward calculation.

Lemma 3 *The functions $\mu_i(n)$ satisfy to the following closed system*

$$\begin{aligned} \mu_1(n+1) &= \mu_1(n) + [\alpha_{12}(\mu_2(n) - \mu_1(n)) + v_1] \gamma_{N_1, N_2} + \alpha_{12}(v_2 - v_1) \gamma_{N_1, N_2}^2 \\ \mu_2(n+1) &= \mu_2(n) + [\alpha_{21}(\mu_1(n) - \mu_2(n)) + v_2] \gamma_{N_1, N_2} + \alpha_{21}(v_1 - v_2) \gamma_{N_1, N_2}^2 \end{aligned}$$

For l_{12} the equation is also linear and closed. Namely,

$$\begin{aligned} l_{12}(n+1) &= l_{12}(n) + [-(\alpha_{12} + \alpha_{21})l_{12}(n) + (v_1 - v_2)] \gamma_{N_1, N_2} + (\alpha_{12} + \alpha_{21})(v_2 - v_1) \gamma_{N_1, N_2}^2 \\ &= l_{12}(n) [1 - \gamma_{N_1, N_2}(\alpha_{12} + \alpha_{21})] + \gamma_{N_1, N_2}(v_1 - v_2) [1 - \gamma_{N_1, N_2}(\alpha_{12} + \alpha_{21})], \end{aligned}$$

and thus we get

$$l_{12}(n) = l_{12}(0)R^n + \frac{v_1 - v_2}{\alpha_{12} + \alpha_{21}}(1 - R^n)R, \quad (2)$$

$$= \frac{v_1 - v_2}{\alpha_{12} + \alpha_{21}}R + \left(l_{12}(0) - \frac{v_1 - v_2}{\alpha_{12} + \alpha_{21}}R \right) R^n \quad (3)$$

where $R = 1 - \gamma_{N_1, N_2}(\alpha_{12} + \alpha_{21})$.

The variables $S(n) = \alpha_{21}\mu_1(n) + \alpha_{12}\mu_2(n)$ also satisfy the recurrent equations

$$S(n+1) = S(n) + \gamma_{N_1, N_2}[\alpha_{21}v_1 + \alpha_{12}v_2]$$

Thus

$$S(n) = S(0) + n \frac{\alpha_{21}v_1 + \alpha_{12}v_2}{N_1\alpha_{12} + N_2\alpha_{21}} = S(0) + (n\gamma_{N_1, N_2}) \cdot (\alpha_{21}v_1 + \alpha_{12}v_2) \quad (4)$$

From (3) and (4) the statement of Theorem 1 follows.

Empirical variances We would like to get closed recurrent equations for $d_i(n)$.

Lemma 4 *The following identity holds*

$$\begin{aligned} &\mathbb{E} \left(D_1(n+1) \mid \left(x_i^{(1)}(t), x_j^{(2)}(t), i = 1, \dots, N_1, j = 1, \dots, N_2 \right), t \leq \tau_n \right) = \\ &= D_1(n) + \alpha_{12}\gamma_{N_1, N_2} \left[\frac{N_1 - 1}{N_1} (D_2(n) - D_1(n) + (X_1(n) - X_2(n))^2) - \frac{2}{N_1} D_1(n) + \right. \end{aligned}$$

$$+2 \frac{N_1 - 1}{N_1} \left(\gamma_{N_1, N_2} (v_1 - v_2) (X_1(n) - X_2(n)) + \gamma_{N_1, N_2}^2 (v_1 - v_2)^2 \right) \Big]$$

and a similar identity for $\mathbf{E}(D_2(n+1) \mid \dots)$ can be obtained by a simple exchange of indices $1 \leftrightarrow 2$.

Proof of this lemma is a straightforward calculation. Taking expectations in the above formulae we see that in the equations for $d_i(n)$ the term $r(n)$ is also involved.

Consider the vector $w(n) = (d_1(n), d_2(n), r(n))^T$. We have

$$w(n+1) = Aw(n) + f(n) + g, \quad (5)$$

where A is a (3×3) -matrix, not depending on n , $f(n)$ is a bounded vector function of n , g is a constant vector

$$A = E + \gamma_{N_1, N_2} B, \quad (6)$$

$$B = B_1 + B_2 = \begin{pmatrix} -\alpha_{12} & \alpha_{12} & \alpha_{12} \\ \alpha_{21} & -\alpha_{21} & \alpha_{21} \\ 0 & 0 & -2(\alpha_{12} + \alpha_{21}) \end{pmatrix} + \begin{pmatrix} -\alpha_{12}/N_1 & -\alpha_{12}/N_1 & -\alpha_{12}/N_1 \\ -\alpha_{21}/N_2 & -\alpha_{21}/N_2 & -\alpha_{21}/N_2 \\ \frac{\alpha_{12}}{N_1} + \frac{\alpha_{21}}{N_2} & \frac{\alpha_{12}}{N_1} + \frac{\alpha_{21}}{N_2} & \frac{\alpha_{12}}{N_1} + \frac{\alpha_{21}}{N_2} \end{pmatrix},$$

$$f(n) = 2 \gamma_{N_1, N_2} (v_1 - v_2) l_{12}(n) \vec{q}_{N_1, N_2}, \quad \vec{q}_{N_1, N_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \gamma_{N_1, N_2} \begin{pmatrix} \alpha_{12} - \alpha_{12}/N_1 \\ \alpha_{21} - \alpha_{21}/N_2 \\ -(\alpha_{12} + \alpha_{21}) \end{pmatrix}, \quad (7)$$

$$g = \gamma_{N_1, N_2}^2 (v_1 - v_2)^2 \begin{pmatrix} \alpha_{12} \gamma_{N_1, N_2} \\ \alpha_{21} \gamma_{N_1, N_2} \\ 2 \end{pmatrix}.$$

Note that $A, f(n), g$ depend on N_1, N_2 and on other parameters of the model. Note that for sufficiently large N_1, N_2 (if α_{ij} are fixed) all components of the vector \vec{q}_{N_1, N_2} are positive. Note also that the matrix elements of A are all positive for sufficiently large N_1, N_2 . Thus Perron-Frobenius theory is applicable.

Spectral properties of the matrix A It is easy to check that the matrix B_1 has three distinct eigenvalues $\lambda_1 = -(\alpha_{12} + \alpha_{21})$, $\lambda_2 = 0$, $\lambda_3 = -2(\alpha_{12} + \alpha_{21})$. We will study asymptotic behavior of the model for the case when $N_1 = c_1 N$, $N_2 = c_2 N$ and $N \rightarrow \infty$. Denote $\Delta := c_1 \alpha_{12} + c_2 \alpha_{21}$, that is $\gamma_{N_1, N_2} = (N \Delta)^{-1}$.

For large N_i the matrix B is a small perturbation of the matrix B_1

$$B = B_1 + \frac{1}{N} B_{2,k} = B_1 + \frac{1}{N} \begin{pmatrix} -\alpha_{12}/c_1 & -\alpha_{12}/c_1 & -\alpha_{12}/c_1 \\ -\alpha_{21}/c_2 & -\alpha_{21}/c_2 & -\alpha_{21}/c_2 \\ \frac{\alpha_{12}}{c_1} + \frac{\alpha_{21}}{c_2} & \frac{\alpha_{12}}{c_1} + \frac{\alpha_{21}}{c_2} & \frac{\alpha_{12}}{c_1} + \frac{\alpha_{21}}{c_2} \end{pmatrix}.$$

We will use perturbation theory to get eigenvalues of B . For $\lambda_1(N)$ and $\lambda_3(N)$ it is sufficient to write

$$\lambda_1(N) = -(\alpha_{12} + \alpha_{21}) + \underline{O}(N^{-1}), \quad \lambda_3(N) = -2(\alpha_{12} + \alpha_{21}) + \underline{O}(N^{-1}), \quad (8)$$

however for $\lambda_2(N)$ we will use the result from [2], that

$$\lambda_2(N) = \frac{1}{N} (\psi' B_{2,k} \phi) + \underline{O}\left(\frac{1}{N^2}\right), \quad (9)$$

where the column vector ϕ is the right eigenvector of B_1 with eigenvalue 0, the row vector ψ' is the left eigenvector of B_1 with eigenvalue 0, and moreover $\psi' \phi = 1$. One can take

$$\phi = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \psi' = \left(1 + \frac{\alpha_{21}}{\alpha_{12}}, 1 + \frac{\alpha_{12}}{\alpha_{21}}, 1\right) / Z, \quad Z = (\alpha_{12} + \alpha_{21}) \left(\frac{1}{\alpha_{12}} + \frac{1}{\alpha_{21}}\right).$$

Substituting these values to (9), we get

$$\lambda_2(N) = -\frac{\varkappa_2}{N} + \underline{O}\left(\frac{1}{N^2}\right), \quad (10)$$

where

$$\varkappa_2 = 2Z^{-1} \left(\frac{\alpha_{21}}{c_1} + \frac{\alpha_{12}}{c_2}\right) \quad (11)$$

Denote $\sigma_1(N)$, $\sigma_1(N)$, $\sigma_1(N)$ the eigenvalues of the matrix A . From (6) and (8)–(9) we have the following assertion.

Lemma 5 *The eigenvalues of A are*

$$\begin{aligned} \sigma_1(N) &= 1 - \frac{b_1}{N} + \underline{O}\left(\frac{1}{N^2}\right), & \sigma_3(N) &= 1 - \frac{b_3}{N} + \underline{O}\left(\frac{1}{N^2}\right), \\ \sigma_2(N) &= 1 - \frac{b_2}{N^2} + \underline{O}\left(\frac{1}{N^3}\right) \end{aligned} \quad (12)$$

for some positive constants b_1, b_2, b_3 .

We will need also the eigenvectors of A , which we denote correspondingly by $e_1^{(N)}, e_2^{(N)}, e_3^{(N)}$. It is clear that they are also the eigenvectors of the matrix $B_1 + \frac{1}{N} B_{2,k}$. Using the perturbation theory [2] we conclude that $e_1^{(N)}, e_2^{(N)}, e_3^{(N)}$ are small perturbations of the eigenvectors e_1, e_2, e_3 of the matrix B_1 . Thus, calculating e_1, e_2, e_3 , we get

$$e_1^{(N)} = \begin{pmatrix} -\alpha_{12} \\ \alpha_{21} \\ 0 \end{pmatrix} + \underline{O}\left(\frac{1}{N}\right), \quad e_2^{(N)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \underline{O}\left(\frac{1}{N}\right), \quad e_3^{(N)} = \begin{pmatrix} -\alpha_{12}^2 \\ -\alpha_{21}^2 \\ (\alpha_{12} + \alpha_{21})^2 \end{pmatrix} + \underline{O}\left(\frac{1}{N}\right).$$

It is clear that with (8) and (10) it is not difficult to find explicitly the constants b_i . We will need only b_2 :

$$b_2 = \frac{\varkappa_2}{\Delta} = \frac{2}{Z} \cdot \frac{\frac{\alpha_{12}}{c_1} + \frac{\alpha_{21}}{c_2}}{c_1 \alpha_{12} + c_2 \alpha_{21}}. \quad (13)$$

Some lemmas on the asymptotic behaviour The solution of the equation (5) can be uniquely written as

$$w(n) = A^n w(0) + \sum_{j=1}^n A^{j-1} f(n-j) + (1-A)^{-1}(1-A^n)g. \quad (14)$$

The following result shows that the first and last terms in (14) do not influence the asymptotics of $w(n)$.

Lemma 6 *The following estimates hold uniformly in n and N*

$$\|A^n w(0)\| \leq Const, \quad \|(1-A)^{-1}(1-A^n)g\| \leq Const.$$

Proof. Using the basis of eigenvectors of A we can write $w(0) = \sum_{i=1}^3 k_{w,i} e_i^{(N)}$. Then

$A^n w(0) = \sum_{i=1}^3 k_{w,i} (\sigma_i(N))^n e_i^{(N)}$. Note that $\sup_N \|e_i^{(N)}\| < \infty$. Moreover, $|\sigma_i(N)| < 1$, starting from some N . Then the first estimate follows. To get the second estimate we write

$$g = \gamma_{N_1, N_2}^2 \sum_{i=1}^3 k_{g,i}^\circ(N) e_i^{(N)}$$

with the coefficients $k_{g,i}^\circ(N)$, bounded in N . Apply the operator $(1-A)^{-1}(1-A^n)$ to the latter expansion and note that by Lemma 5

$$\gamma_{N_1, N_2}^2 \frac{1 - (\sigma_i(N))^n}{1 - \sigma_i(N)} \leq \frac{Const}{N} \quad i = 1, 3,$$

and

$$\gamma_{N_1, N_2}^2 \frac{1 - (\sigma_2(N))^n}{1 - \sigma_2(N)} \leq Const$$

Then we get the estimate. ■

Now we will analyze the second term in (14)

$$V_N(n) := \sum_{j=1}^n A^{j-1} f(n-j),$$

Note that the vector function $f(n)$, defined by the formula (7), is known explicitly with the formula (2).

Let $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$ are such that $(0, 0, 1)^T = \sum_{i=1}^3 \xi_i e_i$. Then we have immediately that

$$\xi_1 = \frac{\alpha_{21} - \alpha_{12}}{(\alpha_{12} + \alpha_{21})^2}, \quad \xi_2 = \frac{\alpha_{12}\alpha_{21}}{(\alpha_{12} + \alpha_{21})^2}, \quad \xi_3 = \frac{1}{(\alpha_{12} + \alpha_{21})^2}.$$

If $\xi_i(N) \in \mathbb{R}$ is the coefficient in the expansion $\vec{q}_{N_1, N_2} = \sum_{i=1}^3 \xi_i(N) e_i^{(N)}$, then obviously

$$\xi_i(N) = \xi_i + \underline{O}(N^{-1}), \quad i = 1, 2, 3. \quad (15)$$

We have then $V_N(n) = \sum_{i=1}^3 \xi_i(N) \left(2 \gamma_{N_1, N_2} (v_1 - v_2) \sum_{j=1}^n l_{12}(n-j) (\sigma_i(N))^{j-1} \right) e_i^{(N)}$. By (15), and neglecting $\underline{O}(N^{-1})$, we have that for $N \rightarrow \infty$ the asymptotics of $V_N(n)$ coincides with the asymptotics of the sum

$$V_N^1(n) := \sum_{i=1}^3 \xi_i \left(2 \gamma_{N_1, N_2} (v_1 - v_2) \sum_{j=1}^n l_{12}(n-j) (\sigma_i(N))^{j-1} \right) e_i^{(N)}. \quad (16)$$

Lemma 7 *For $n = N\theta(N)$, where $\theta(N) \rightarrow +\infty$, the asymptotics of $V_N^1(n)$ is defined by the second term, that is the first and third are small with respect to the second.*

Remind that $\sigma_2(N)$ is the maximal eigenvalue of the positive matrix A . Then this corresponds to Perron-Frobenius theory.

Proof. Consider the formula (2). If we assume, that $l_{12}(0) < 0$, then from $v_1 < v_2$ it follows that each coordinate of $f(n)$ is positive for all n (we use it below). Moreover, from (2) one can get that there exist constants $C_1 > C_2 > 0$, which do not depend on N and n , such that

$$0 < C_2 < (v_1 - v_2) l_{12}(n) < C_1 \quad \forall n, N \quad (17)$$

Thus, the coefficient of $e_2^{(N)}$ in the sum (16) is positive and can be *estimated from below* as

$$2\xi_2 C_2 \gamma_{N_1, N_2} \sum_{j=1}^n (\sigma_2(N))^{j-1} = 2\xi_2 C_2 \gamma_{N_1, N_2} \frac{1 - (\sigma_2(N))^n}{1 - \sigma_2(N)}$$

Similarly, for $i = 1, 3$ the absolute values of the coefficients of $e_i^{(N)}$ in the sum (16) can be *estimated from above* as

$$2\xi_i C_1 \gamma_{N_1, N_2} \sum_{j=1}^n (\sigma_i(N))^{j-1} = 2\xi_i C_1 \gamma_{N_1, N_2} \frac{1 - (\sigma_i(N))^n}{1 - \sigma_i(N)}$$

Thus, to end the proof of the lemma it is sufficient to compare the asymptotics of the following three functions

$$\frac{1 - (\sigma_1(N))^{N\theta(N)}}{1 - \sigma_1(N)}, \quad \frac{1 - (\sigma_2(N))^{N\theta(N)}}{1 - \sigma_2(N)}, \quad \frac{1 - (\sigma_3(N))^{N\theta(N)}}{1 - \sigma_3(N)}$$

and to show that the first and the third are small with respect to the second. It is convenient to consider separately two cases: **a)** $\theta(N) \rightarrow \infty$, $\theta(N)/N \rightarrow 0$, **b)** $\theta(N) \geq cN$, and use Lemma 5. We omit these details. ■

Asymptotic behavior of expectations of empirical variances Now we are ready to study asymptotics of the functions R_1 and R_2 and to prove Theorem 2. Remind that the intervals between jumps of the embedded chain have exponential distribution with the mean $\gamma_{N_1, N_2} = (N\Delta)^{-1}$, then for large N the connection between discrete time n of the embedded chain and absolute time t is $n \sim t/\gamma_{N_1, N_2} = (N\Delta)t$. Thus we can take $d_i((N\Delta)t)$ instead of $R_i(t)$. Remind also that d_1 and d_2 are the first and second components of the vector w correspondingly.

Now we can use the lemma 7, which shows that, as $t(N) \rightarrow \infty$, the asymptotics of $w((N\Delta)t(N))$ coincides with the asymptotics of the vector

$$\xi_2 \left(2 \gamma_{N_1, N_2} (v_1 - v_2) \sum_{j=1}^{(N\Delta)t(N)} l_{12}((N\Delta)t(N) - j) (\sigma_2(N))^{j-1} \right) e_2^{(N)}.$$

As $e_2^{(N)} = (1, 1, 0)^T + \underline{O}(N^{-1})$, then

$$d_i((N\Delta)t) \sim 2 \xi_2 \gamma_{N_1, N_2} (v_1 - v_2) \sum_{j=1}^{(N\Delta)t(N)} l_{12}((N\Delta)t(N) - j) (\sigma_2(N))^{j-1}.$$

To find the asymptotics of this expression, we use the representation (3), which gives

$$l_{12}(n) = C'_{1,N} + C'_{2,N} R^n, \quad C'_{1,N} \rightarrow \frac{v_1 - v_2}{\alpha_{12} + \alpha_{21}}, \quad C'_{2,N} \rightarrow C'_{2,\infty} \quad (N \rightarrow \infty).$$

Note that the following estimate holds uniformly in N and n

$$\begin{aligned} \left| \gamma_{N_1, N_2} \sum_{j=1}^n C'_{2,N} R^{n-j} (\sigma_2(N))^{j-1} \right| &\leq (N\Delta)^{-1} |C'_{2,N}| \sum_{k=1}^{\infty} R^k = (N\Delta)^{-1} |C'_{2,N}| \cdot \frac{1}{1-R} \\ &\leq \text{Const}, \end{aligned}$$

since $1 - R = \gamma_{N_1, N_2}(\alpha_{12} + \alpha_{21})$. Consider now the asymptotics of the following expression

$$2 \xi_2 \gamma_{N_1, N_2} (v_1 - v_2) \sum_{j=1}^{(N\Delta)t(N)} \frac{v_1 - v_2}{\alpha_{12} + \alpha_{21}} (\sigma_2(N))^{j-1} = 2 \xi_2 \frac{1}{N\Delta} \cdot \frac{(v_1 - v_2)^2}{\alpha_{12} + \alpha_{21}} \cdot \frac{1 - (\sigma_2(N))^{(N\Delta)t(N)}}{1 - \sigma_2(N)}.$$

By (12) and (13) $\sigma_2(N) = 1 - \frac{(\kappa_2/\Delta)}{N^2} + \underline{O}(N^{-3})$ and thus the problem is reduced to the study of asymptotics of

$$\frac{2 \xi_2 (v_1 - v_2)^2}{\kappa_2 (\alpha_{12} + \alpha_{21})} N \left(1 - \left(1 - \frac{(\kappa_2/\Delta)}{N^2} \right)^{(N\Delta)t(N)} \right).$$

Now the theorem easily follows.

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